RINGS WITH ZERO RIGHT AND LEFT SINGULAR IDEALS(1)

BY
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The work of this paper was done independently of and at the same time as that of Utumi published in [1] and [2]. Although our results overlap those of Utumi, our approach to the problem at hand is quite different. Thus, it might be worthwhile to record another approach to the determination of the structure of a ring of the type stated in the title.

Our principle result is that a ring R is finite-dimensional, stable and irreducible iff it contains a subring of all matrices of the form

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & 0 & \cdots & 0 \\ c & b_2 & \cdots & b_n \end{bmatrix},$$

where $a_i \in F_1$, $c \in F_{n1}$, and $b_i \in F_n$, and F_1 is a right and F_n a left Ore subring of a division ring F associated with R and F_{n1} is an (F_n, F_1)-module contained in F and containing F_1 and F_n.

1. **Introduction.** If R is a ring, then $L_r(R)$, $L_r^\Delta(R)$, and R_r^Δ designate the lattice of right ideals, the lattice of large right ideals, and the right singular ideal, respectively, of R. We recall that $A \in L_r^\Delta(R)$ iff $A \in L_r(R)$ and $A \cap B \neq 0$ for every nonzero $B \in L_r(R)$; and $a \in R_r^\Delta$ iff $a^r \in L_r^\Delta(R)$, where a^r is the right annihilator of a in R. If $R_r^\Delta = 0$, we can define a closure operation $*$ on $L_r(R)$ as follows: A^* is the maximal essential extension of A in $L_r(R)$. In this case, the set $L_r^*(R)$ of all closed right ideals of R is a complete complemented modular lattice. The lattice $L_r^*(R)$ has a dimension function, $\dim_r$, defined on it, with $\dim_r(A)$ being the length of the longest chain in the interval $[0, A]$. If $L_r^*(R)$ is either Noetherian or Artinian, then $\dim_r(R) < \infty$ (see [3]). It is immediate that $\dim_r(R) = 1$ iff R is a right Ore domain.

A ring Q is called a *right quotient ring* of R, and we write $R \leqq_r Q$, iff $qR \cap R \neq 0$ for each nonzero $q \in Q$. If $R \leqq_r Q$, then $Q_r^\Delta = 0$ iff $R_r^\Delta = 0$; and if $R_r^\Delta = 0$ then $L_r^*(R) \cong L_r^*(Q)$ under the correspondence $A \to A \cap R$, $A \in L_r^*(Q)$ [3, 2.6].

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We shall replace the letter "r" above by "l" to designate the corresponding left property of R. It will be assumed throughout this paper that

$$R_r^\Delta = 0 \quad \text{and} \quad R_l^\Delta = 0$$

for each ring R under discussion. Many of the results of the paper are stated only for right or left properties of a ring, although they clearly hold for right and left properties.

If a lattice L has minimal nonzero elements, called atoms, and if each nonzero $A \in L$ contains an atom, then L is called *atomic*. It is possible for one of the lattices $L_r^*(R)$ and $L_l^*(R)$ to be atomic and the other to contain no atoms at all. Such is the case, for example, if R is a right Ore domain but not a left Ore domain. It is also possible for both $L_r^*(R)$ and $L^*(R)$ to be atomic and still be quite different, as 1.2 and 1.3 below show. If $L_r^*(R)$ is atomic then it contains maximal elements called *co-atoms*. Clearly $A \in L_r^*(R)$ is a co-atom iff each complement of A is an atom.

The union in $L_r(R)$ of the set of all atoms of $L_r^*(R)$ is called the *right base* of R and is designated by R_r^0. If $L_r^*(R)$ is atomic then it is easily shown that R_r^0 is an ideal of R and

$$(1.1) \qquad (R_r^0)^l = 0.$$

The lattice $L_r^*(R)$ has a *center* $C_r^*(R)$ consisting of those elements having unique complements. It was shown in [4, p. 541] that $C_r^*(R) = \{A \mid A \text{ ideal of } R, A \cap A^l = 0, A = A^{ll}\}$. Since $A^l \in L_l^*(R)$ for each subset A of R, evidently $C_r^*(R) \subset L_l^*(R)$. The ring R is called *right irreducible* iff $C_r^*(R) = \{0, R\}$, and *irreducible* iff it is both right and left irreducible.

1.2. Example. Let $K = \{at + b \mid a,\ b \in Z_2,\ t^2 = t + 1\} = GF(2^2)$ and

$$R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \;\middle|\; a \in Z_2,\ b,\ c \in K \right\},$$

a subring of K_2. Since $R \leqq_l K_2$, $\dim_l(R) = 2$. On the other hand, R may b shown to be isomorphic to the subring S of $(Z_2)_3$ of all matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ b_1 & d_1 & d_2 \\ b_2 & d_2 & d_1 + d_2 \end{bmatrix}$$

the matrix above corresponding to the matrix

$$\begin{pmatrix} a & 0 \\ b_1 + b_2 t & d_1 + d_2 t \end{pmatrix}$$

of R. Since $S \leqq_r (Z_2)_3$, $\dim_r(R) = \dim_r(S) = 3$. Furthermore, R is irreducible since K_2 and $(Z_2)_3$ are. It may be shown that $R_l^0 = R$ and $R_r^0 = Z_2 e_{11} + K e_{21}$.

1.3. EXAMPLE. Let F be a field, $\{e_{ij} \mid i,j = 1,2,\cdots\}$ be a set of matrix units, $x = \sum_{i=2}^{\infty} e_{i+1,i}$, $S = xF[x]$, T_n be the weak direct sum $T_n = \sum_{i=n}^{\infty} Fe_{i1}$, and $R = S + T_1$. We shall not bother with the details but it may be shown that $R_r^{\Delta} = R_l^{\Delta} = 0$, $R_r^0 = T_1$, $R_l^0 = R$, $\dim_r(R) = \infty$, $\dim_l(R) = 2$, and R is irreducible.

The ring R is called *right atomic* if $L_r^*(R)$ is atomic, and atomic if it is both right and left atomic.

1.4. THEOREM. *If R is an atomic ring and $A \in L_l^*(R)$ is an atom, then either $A \subset R_r^0$ or $A \subset (R_r^0)^r$. If A^r is a co-atom of $L_r^*(R)$ then $A \subset R_r^0$, and if $A \not\subset (R_r^0)^r$ then A^r is a co-atom of $L_r^*(R)$.*

Proof. Since A is an atom, $A^r = a^r$ for each nonzero $a \in A$. If A^r is a co-atom of $L_r^*(R)$, then each $a \in A$ is contained in an atom of $L_r^*(R)$ by [4, 6.9] and therefore $A \subset R_r^0$. On the other hand, if $A \not\subset (R_r^0)^r$ then $BA \neq 0$ for some atom $B \in L_r^*(R)$ and $ba \neq 0$ for some $b \in B$ and $a \in A$. Since $ba \in B \cap A$ and $(ba)^r$ is a co-atom of $L_r^*(R)$ by [4, 6.9], A^r is a co-atom of $L_r^*(R)$. This proves 1.4.

If $(R_r^0)^r \subset R_r^0$ then, by 1.4, $R_l^0 \subset R_r^0$ and $(R_l^0)^r \supset (R_r^0)^r$. Since $(R_l^0)^r = 0$ by (1.1), we conclude that $(R_r^0)^r \subset R_r^0$ iff $(R_r^0)^r = 0$. Also, $R_l^0 \subset R_r^0 + (R_r^0)^r$ by 1.4, and therefore $(R_r^0)^r \cap (R_r^0)^{rr} = 0$. This proves the following theorem.

1.5. THEOREM. *If R is an atomic ring then $(R_r^0)^r = 0$ iff $(R_r^0)^r \subset R_r^0$. In any case, $(R_r^0)^r \cap (R_r^0)^{rr} = 0$.*

If R is an atomic ring such that $\dim_r(R) = 2$, then either $(R_r^0)^r = 0$ or $(R_r^0)^r$ is an atom of $L_r^*(R)$. Since the latter possibility contradicts 1.5, evidently $(R_r^0)^r = 0$. Thus, 1.2, in which $\dim_r(R) = 3$ and $(R_r^0)^r = e_{22}R$, is in some sense the simplest possible one having $(R_r^0)^r \neq 0$.

1.6. THEOREM. *If R is a right atomic ring and $A, B \in L_r^*(R)$ such that A is an atom and $BA \neq 0$, then $B^r \subset A^r$.*

Proof. Since $BA \subset B$, $(BA)^r \supset B^r$. By assumption $bA \neq 0$ for some $b \in B$. Hence, $B^r \cap A = 0$ and $(BA)^r = A^r$. This proves 1.6.

2. **Stable rings.** A ring R is called *right stable* iff R is right atomic and $(R_r^0)^r = 0$, and *stable* iff R is both right and left stable. As Examples 1.2 and 1.3 show a ring can be right or left stable without being stable. It is easily seen that a semiprime atomic ring is stable.

2.1. THEOREM. *If R is a right stable ring and $R \leq_r Q$, then Q is also a right stable ring.*

Proof. If $q \in (Q_r^0)^r$, $q \neq 0$, then $qa = b \neq 0$ for some $a, b \in R$. Since $Aq = 0$ for each atom $A \in L_r^*(Q)$, also $Bq = 0$ for each atom $B \in L_r^*(R)$. Hence, $b \in (R_r^0)^r$ contrary to the assumption that $(R_r^0)^r = 0$. This contradiction proves 2.1.

2.2. Theorem. *If R is an atomic, right stable ring then A^r is a co-atom of $L_r^*(R)$ for each atom $A \in L_l^*(R)$ and B^r is either 0 or an atom of $L_r^*(R)$ for each co-atom $B \in L_l^*(R)$. In case B^r is an atom, $B^{rl} = B$.*

Proof. The first part follows directly from 1.4. Let B be a co-atom of $L_l^*(R)$ such that $B^r \neq 0$. We shall prove that B^r is an atom of $L_r^*(R)$ by showing that $xR \cap yR \neq 0$ for all nonzero $x, y \in B^r$. So let x and y be any nonzero elements of B^r. Clearly $x^l = y^l = B$. If C is a complement of B and $c \in C$, $c \neq 0$, then $cx \neq 0$ and $cy \neq 0$. Since c^r is a co-atom of $L_r^*(R)$, evidently $c \in D$, an atom of $L_r^*(R)$, by [4, 6.9]. Therefore, $cx, cy \in D$ and $cxR \cap cyR \neq 0$. Hence, $cxp = cxq \neq 0$ for some $p, q \in R$ and $c(xp - yq) = 0$. Since $(xp - yq)^l \supset B + C$ and $B + C \in L_l^\Delta(R)$, necessarily $xp - yq = 0$. Thus, $xR \cap yR \neq 0$. Since $B^{rl} \supset B$ and B is a co-atom, $B^{rl} = B$. This proves 2.2.

2.3. Corollary. *If R is a stable ring, then $A^{lr} = A$ and $B^{rl} = B$ for all atoms $A \in L_r^*(R)$ and $B \in L_l^*(R)$.*

That the conclusions of 2.2 need not hold for a nonstable ring is shown by Example 1.3. Thus, S is an atom of $L_l^*(R)$ for which $S^r (= Fe_{11})$ is an atom, but not a co-atom, of $L_r^*(R)$; and T_1 is a co-atom of $L_l^*(R)$ for which $T_1^r (= S + T_2)$ is a co-atom, but not an atom, of $L_r^*(R)$.

2.4. Theorem. *If R is an atomic ring, then R is right stable iff $R_l^0 \subset R_r^0$.*

Proof. If R is right stable, $R_l^0 \subset R_r^0$ by 1.4. Conversely, if $R_l^0 \subset R_r^0$ then $0 = (R_l^0)^r \supset (R_r^0)^r$ by (1.1) and R is right stable.

2.5. Corollary. *An atomic ring R is stable iff $R_r^0 = R_l^0$.*

Let us now examine the relationship between $C_r^*(R)$ and $C_l^*(R)$ for a stable ring R.

2.6. Theorem. *If R is a right stable ring, then $C_r^*(R) \subset C_l^*(R)$.*

Proof. Let $A \in C_r^*(R)$ and $B = A \cap A^r$. Since $B \cap A^l \subset A \cap A^l = 0$, we have $AB = A^l B = 0$. Every atom of $L_r^*(R)$ is contained in either A or its complement A^l; therefore, $B \subset (R_r^0)^r$ and $B = 0$. It follows that $A^r = A^l$ for every $A \in C_r^*(R)$. Since $A^l \in C_r^*(R)$, evidently $A^{lr} = A^{ll}$ and $A^{lr} = A^{rr}$. Hence, $A = A^{ll} = A^{rr}$ and $A \in C_l^*(R)$. This proves 2.6.

2.7. Corollary. *If R is a stable ring, then $C_r^*(R) = C_l^*(R)$.*

2.8. Corollary. *If R is a stable ring, then R is right irreducible iff R is left irreducible.*

The ring R of Example 1.2 is left stable but not right stable. We recall that $\dim_l(R) = 2$ and $\dim_r(R) = 3$. This example illustrates the following theorem.

2.9. THEOREM. *If R is an atomic, right stable ring then* $\dim_r(R) \leqq \dim_l(R)$. *If, in addition,* $\dim_r(R) < \infty$ *then* $\dim_r(R) = \dim_l(R)$ *iff R is also left stable.*

Proof. If S is the set of all atoms of $L_l^*(R)$, then $T = \{A^r \mid A \in S\}$ is the set of all co-atoms of $L_r^*(R)$ by 2.2. Clearly $\bigcap_{A \in S} A^r = 0$. Therefore, if $\dim_r(R) = n < \infty$, there exist $A_1, \cdots, A_n \in S$ such that $\bigcap_{i=1}^n A_i^r = 0$. Since $\dim_r(A_i^r) = n - 1$ and $\dim_r(A_i^r \cap C) \geqq \dim_r(C) - 1$ for each $C \in L_r^*(R)$, necessarily $\dim_r(\bigcap_{i=1}^k A_i^r) = n - k$, $k = 1, \cdots, n$, in order that $\bigcap_{i=1}^n A_i^r = 0$. Now $(A_1 \cup \cdots \cup A_k)^r = (A_1 + \cdots + A_k)^r = A_1^r \cap \cdots \cap A_k^r$ by the definition of union in $L_l^*(R)$. Therefore, $A_{k+1} \not\subset A_1 \cup \cdots \cup A_k$ for $k = 1, \cdots, n-1$. Thus, $\{A_1, \cdots, A_n\}$ is an independent set of atoms of $L_l^*(R)$, and $\dim_l(R) \geqq \dim_r(R)$.

If $\dim_l(R) = m < \infty$ and $\{A_1, \cdots, A_m\}$ is an independent set of atoms of $L_l^*(R)$, then $\bigcap_{i=1}^m A_i^r = 0$. It follows by lattice theory that $\dim_r(R) \leqq m$. Therefore, if $\dim_r(R) = \infty$ then also $\dim_l(R) = \infty$. Clearly we have proved that $\dim_r(R) = \dim_l(R)$ if R is stable.

Finally, let us assume that R is atomic and right stable, and that $\dim_r(R) = \dim_l(R) = n < \infty$. Let us choose the independent set of atoms of $L_l^*(R)$, $\{A_1, \cdots, A_n\}$, as we did above, and define $B_i = \bigcap_{j \neq i} A_j^r$. Since $(B_1 \cup \cdots \cup B_k) \subset A_{k+1}^r \cap \cdots \cap A_n^r$ and $B_{k+1} \cap A_{k+1}^r = 0$, evidently $(B_1 \cup \cdots \cup B_k) \cap B_{k+1} = 0$, $k = 1, \cdots, n-1$. Hence, $\{B_1, \cdots, B_n\}$ is an independent set of atoms of $L_r^*(R)$. Clearly B^l is a co-atom of $L_l^*(R)$ for each i. Therefore, by 1.4, $B_i \subset R_l^0$ for each i and $0 = R^l = (B_1 \cap \cdots \cap B_n)^l \supset (R_l^0)^l$. Thus, R is left stable. This proves 2.9.

2.10. THEOREM. *If R is a stable ring and $A, B \in L_r^*(R)$ are such that A is an atom and $B^{lr} = B$, then $(A \cup B)^{lr} = A \cup B$.*

Proof. The theorem obviously is true if $B = 0$ or $A \subset B$, so let us assume henceforth that $B \neq 0$ and $A \cap B = 0$. Since $(A \cup B)^l = A^l \cap B^l$ and $(A \cup B)^{lr} \supset A \cup B$, we can prove 2.10 by showing that every atom of $L_r^*(R)$ contained in $(A^l \cap B^l)^r$ is also contained in $A \cup B$. So let $C \subset (A^l \cap B^l)^r$, C an atom of $L_r^*(R)$. If $C \subset A$ or $C \subset B$, then $C \subset A \cup B$ and we are through. Thus, we might as well assume that $C \cap A = C \cap B = 0$. Since $B^l \not\subset A^l$, there exists an atom $E \in L_l^*(R)$ such that $E \subset B^l$ and $E \cap A^l = 0$. Clearly $E \cup A^l = R$, so that $E \cup (A^l \cap B^l) = (E \cup A^l) \cap B^l = B^l$ by the modular law. Hence, $E^r \cap (A^l \cap B^l)^r = B$. Since E^r is a co-atom of $L_r^*(R)$, we have $E^r \cap (A \cup C) \neq 0$. Now $A \cup C \subset (A^l \cap B^l)^r$, and therefore $E^r \cap (A \cup C) \subset E^r \cap (A^l \cap B^l)^r = B$. Hence, $(A \cup C) \cap B \neq 0$ and $C \subset A \cup B$. This proves 2.10.

Atoms A and B of $L_r^*(R)$ are called *perspective*, $A \sim B$, iff they have a common complement. The union in $L_r^*(R)$ of all atoms perspective to an atom A is an atom of $C_r^*(R)$ according to [4, 6.12]. We shall use the fact that $A \sim B$ iff there exist nonzero $a \in A$ and $b \in B$ such that $a^r = b^r$ [4, 6.10].

2.11. Lemma. *If R is an atomic, right stable ring, and A and B are perspective atoms of $L_l^*(R)$, then $AR \cap BR \neq 0$.*

Proof. Since $A \sim B$, there exist nonzero $a \in A$ and $b \in B$ such that $a^l = b^l$. Since a^l is a co-atom of $L_l^*(R)$, a^{lr} is an atom of $L^s(R)$ by 2.2. Clearly $a, b \in a^{lr}$ and therefore $aR \cap bR \neq 0$. This proves 2.11.

2.12. Theorem. *If R is an atomic, right stable, irreducible ring then $S \cap T \neq 0$ for every pair S, T of nonzero ideals of R.*

Proof. Let A and B be atoms of $L_l^*(R)$ such that $A \subset S^*$ and $B \subset T^*$. Since R is irreducible, $A \sim B$ and $AR \cap BR \neq 0$ by 2.11. Now S^* and T^* are ideals of R by [4, 4.2], and therefore $S^* \cap T^* \neq 0$. Hence $S \cap T \neq 0$ and 2.12 is proved.

If ring R is atomic, right stable, and irreducible, and if P is the set of all atoms of $L_r^*(R)$, then $\bigcap_{A \in P} A^r = 0$. If, in addition, $\dim_r(R) < \infty$ then $\bigcap_{i=1}^n A_i^r = 0$ for some finite subset $\{A_1, \cdots, A_n\}$ of P. Since each A_i^r is an ideal of R, some $A_i^r = 0$ by 2.12. This proves the following theorem.

2.13. Theorem. *If R is an atomic, right stable, irreducible ring such that $\dim_r(R) < \infty$, then $A^r = 0$ for some atom $A \in L_r^*(R)$.*

Ring R of 1.3 is an example of an atomic, left stable, and irreducible ring such that $\dim_l(R) < \infty$. For this ring, T_1 is an atom of $L_l^*(R)$ such that $T_1^l = 0$. We note that $A^r \neq 0$ for every atom $A \in L_r^*(R)$.

If R is a ring satisfying the conditions of 2.13, and if R is not left stable, then $(R_l^0)^{ll} = 0$ in view of 1.5.

3. **Finite-dimensional rings.** For convenience, let us call a finite-dimensional stable ring an FS-*ring*, and an irreducible FS-ring an FSI-*ring*. An FS-ring R of dimension 1 is an Ore domain. That is, $R \subset D$, a division ring, and $RR^{-1} = R^{-1}R = D$, where $AB^{-1} = \{ab^{-1} \mid a \in A,\ b \in B,\ b \neq 0\}$.

3.1. Theorem. *If R is an FS-ring, then the lattices $L_r^*(R)$ and $L_l^*(R)$ are dual isomorphic under the correspondence $A \to A^l$, $A \in L_r^*(R)$.*

Proof. Since each $A \in L_r^*(R)$ is a finite union of atoms, $A^{lr} = A$ by 2.10. Similarly, $B^{rl} = B$ for each $B \in L_l^*(R)$. This proves 3.1.

Every ring R contains a maximal nil ideal $N = N(R)$. If $R_r^\Delta = 0$ and $\dim_r(R) < \infty$, then the maximal right quotient ring of R is a finite direct sum of total matrix rings over division rings [3, 4.3] and N is nilpotent by [5, Theorem 6]. In particular, if R is an FSI-ring then either $N = 0$ and R is a prime ring or N has a finite nonzero index of nilpotency $i(N)$. Of course, N is the prime radical of R in this case.

3.2. Theorem. *If R is an FSI-ring of dimension $n > 1$, then there exist atoms $A \in L_r^*(R)$ and $B \in L_l^*(R)$ such that $A^r = B^l = BA = 0$.*

Proof. If R is prime, then we may select A and B as any atoms for which $BA = 0$. If R is not prime, then there exist atoms A and B such that $A^r = B^l = 0$ by 2.13. Let $N = N(R)$ and $i = i(N)$. Since $AN^{i-1} \neq 0$, $NB \neq 0$, and $(NB)(AN^{i-1}) = 0$, evidently $BA = 0$. This proves 3.2.

If R is an FSI-ring of dimension $n > 1$, then by [3, 4.1] $R \leqq_r Q$, where Q is the full ring of linear transformations of an n-dimensional vector space over a division ring. Actually, $R \leqq_l Q$ also as we shall now show (along the lines of Utumi in [1] and [2]). If $e \in Q$, e an idempotent, and $A = eQ \cap R$, $A' = (1-e)Q \cap R$, $B = Qe \cap R$, and $B' = Q(1-e) \cap R$, then $A^l = B'$ and $A'^l = B$ in R because of 3.1. Since $A \cap A' = 0$, $B \cup B' = R$ and therefore $B + B' \in L_l^{\Delta}(R)$. Clearly $0 \neq (B + B')e \subset R$. Now Q is generated by its idempotents [6, Theorem 5.1], and therefore for each $q \in Q$ there exists some $C \in L_l^{\Delta}(R)$ such that $Cq \subset R$. If $J = \{q \in Q \mid q^l \cap R \in L_l^{\Delta}(R)\}$, then J is a right ideal of Q for which $J \cap R = 0$. Since $R \leqq_r Q$, evidently $J = 0$. Hence, $Rq \cap R \neq 0$ for every nonzero $q \in Q$ and $R \leqq_l Q$.

We are now in a position to prove the main theorem of the paper.

3.3. THEOREM. *Let R be a right irreducible ring of finite rank $n > 1$ and let Q be its maximal right quotient ring. Thus, Q is the full ring of linear transformations of an n-dimensional vector space over a division ring F. The ring R is an FSI-ring iff R contains a subring T of the form*

$$T = F_1e_{11} + \cdots + F_1e_{n-1,1} + F_{n1}e_{n1} + F_ne_{n2} + \cdots + F_ne_{nn},$$

where $\{e_{ij} \mid i,j = 1,\cdots,n\} \subset Q$ is a set of matrix units, F_1 and F_n are subrings of F such that $F_1 \leqq_r F$ and $F_n \leqq_l F$, and F_{n1} is an (F_n, F_1)-module contained in F and containing F_1 and F_n.

Proof. Let T be a subring of Q of the stated form and $q = \sum d_{ij}e_{ij} \in Q$, $d_{ij} \in F$, with $d_{km} \neq 0$ for some k, m. By assumption, $d_{im} = a_ib_i^{-1}$ for some $a_i, b_i \in F_1$, $i = 1,\cdots,n$. If $b \in \bigcap_{i=1}^n b_iF_1$, $b \neq 0$, then $be_{m1} \in T$ and $qbe_{m1} = \sum(a_ib_i^{-1}b)e_{i1} \in T$ also. Hence, $qT \cap T \neq 0$. We conclude that $T \leqq_r Q$. Since the conditions on T are right-left symmetric, $T \leqq_l Q$ also. It is clear that $T_r^0 = T_l^0 = T$ and hence that T is an FSI-ring. Therefore, R is an FSI-ring by 2.1.

Assume, conversely, that R is an FSI-ring. Then, $R \leqq_l Q$ also by our remarks above. Let $A \in L_r^*(R)$ and $B \in L_l^*(R)$ be atoms such that $A^r = B^l = BA = 0$, as given in 3.2. We may select a set $\{e_{ij} \mid i,j = 1,\cdots,n\}$ of matrix units of Q such that $A = e_{nn}Q \cap R$ and $B^r = e_{22}Q + \cdots + e_{nn}Q$ in Q [7, p. 52]. Then $B = B^{rl} = Qe_{11} \cap R$ in R. Let $A_i = e_{ii}Q \cap R$ and $B_i = Qe_{ii} \cap R$, $i = 1,\cdots,n$, so that $A = A_n$ and $B = B_1$. We may consider Q to be the ring of all $n \times n$ matrices over F. Then $A_i \cap B_j = F_{ij}e_{ij}$ for some additive subgroups F_{ij} of F. Since $A_n^r = B_1^l = 0$ evidently $F_{ij} \neq 0$ if $i = n$ or $j = 1$.

Since $B_1^l = 0$ and B_1 is a left ideal of R, we have $B_1 \leqq_r R$. Hence, $\{F_{11}e_{11}, \cdots, F_{n1}e_{n1}\}$ is an atomic basis of $L_r^*(B_1)$ and $S \leqq_r Q$, where

$S = F_{11}e_{11} + \cdots + F_{n1}e_{n1}$. Therefore, for each nonzero $d \in F$ and each e_{ij}, we must have $(de_{ij}S) \cap S \neq 0$. It follows that $dF_{j1} \cap F_{i1} \neq 0$ for all i, j, that is,

$$F = F_{i1}F_{j1}^{-1}, \qquad i, j = 1, \cdots, n. \tag{3.4}$$

We may prove similarly, by working with A_n, that $F = F_{ni}^{-1}F_{nj}$ for all i and j.

Next, we observe that $F' = \bigcap_{i=1}^{n} F_{i1} \neq 0$. For if $F'_k = \bigcap_{i=1}^{k} F_{i1}$ and $F'_m \neq 0$ for some $m < n$, and if $c \in F'_m$, $c \neq 0$, then $c = ab^{-1}$ for some $a \in F_{m+1,1}$ and $b \in F_{11}$ by (3.4). Since $F'_k F_{11} \subset F'_k$ for each k and $a = cb$, evidently $a \in F'_{m+1}$. Thus, $F' \neq 0$ by induction. Similarly, if $F'' = \bigcap_{j=1}^{n} F_{nj}$ then $F'' \neq 0$. Let us select nonzero $f_1 \in F'$ and $f_n \in F''$, and then define $F_1 = f_1 F_{11}$, $F_n = F_{nn} f_n$. Clearly F_1 and F_n are subrings of F such that $F_1 \leqq_r F$ and $F_n \leqq_l F$, and also $F_1 \subset F_{n1}$ and $F_n \subset F_{n1}$.

If we let $T = F_1 e_{11} + \cdots + F_1 e_{n-1,1} + F_{n1}e_{n1} + F_n e_{n2} + \cdots + F_n e_{nn}$, then T is a subring of R having the stated properties. This completes the proof of 3.3.

We remark that with some extra work, it is possible to prove 3.3 without first proving that $R \leqq_l Q$. This can be accomplished by proving directly the existence of the atomic bases $\{A_1, \cdots, A_n\}$ and $\{B_1, \cdots, B_n\}$ used in the above proof of 3.3.

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